Selected Solution to Assignment 5

Supplementary Problems

1. (Optional) Let P be a plane given by the equation $ax+by+cz=d$. Show that the distance from a point (x_0, y_0, z_0) to P is given by the formula

$$\frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}} \ .$$

Hint: Treat it as a constrained minimization problem.

Solution. Minimize the function $F(x,y,z)=(x-x_0)^2+(y-y_0)^2+(z-z_0)^2$ subject to the constraint $g(x,y,z)\equiv ax+by+cz-d=0$. By Lagrange multipliers, the minimizer (x,y,z) satisfies $\partial F/\partial x=\lambda\partial g/\partial x, \partial F/\partial y=\lambda\partial g/\partial y, \partial F/\partial z=\lambda\partial g/\partial z$. In other words,

$$x-x_0=\lambda a,\ \ y-y_0=\lambda b\ , z-z_0=\lambda c\ .$$

Plugging this into the constraint,

$$a(x_0+\lambda a)+b(y_0+\lambda b)+c(z_0+\lambda c)=d$$

gives

$$\lambda=\frac{d-(ax_0+by_0+cz_0)}{a^2+b^2+c^2}\ .$$

Using this, the minimizing point is attained at $x=x_0+\lambda a, y=y+0+\lambda b, z=z_0+\lambda c$ and the distance is given by

$$\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}=\sqrt{\lambda^2a^2+\lambda^2b^2+\lambda^2c^2}=\frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}\ .$$

2. Let Ω be a region in space which is symmetric with respect to the xy-plane, that is, $(x,y,z)\in\Omega$ if and only if $(x,y,-z)\in\Omega$. Show that

$$\iiint_\Omega f(x,y,z)\,dV=0\ ,$$

when f is odd in z, that is, $f(x,y,-z)=-f(x,y,z)$ in Ω. You may assume Ω is of the form $\{(x,y,z):\ f_1(x,y)\le z\le f_2(x,y), (x,y)\in D\}$.

Solution. By assumption, $f_1=-f_2\le 0$ and so Ω can be decomposed into the union of Ω_1 and Ω_2 where $\Omega_1=\{(x,y,z):\ -f_2(x,y)\le z\le 0\}$ and $\Omega_2=\{(x,y,z):\ 0\le z\le f_2(x,y)\}$. We have

$$\iiint_\Omega f\,dV=\iiint_{\Omega_1} f\,dV+\iiint_{\Omega_2} f\,dV\ .$$

As

$$\begin{aligned}\iiint_{\Omega_1} f dV &= \iint_D\int_{-f_2(x,y)}^0 f(x,y,z)\,dzdA(x,y)\\ &= \iint_D\int_0^{f_2(x,y)} f(x,y,-z)\,dzdA(x,y)\\ &= -\iint_D\int_0^{f_2(x,y)} f(x,y,z)\,dzdA(x,y)\quad \text{(since } f \text{ is odd in } z)\\ &= -\iiint_{\Omega_2} f\,dV\ ,\end{aligned}$$

and the conclusion follows.